On the solutions of the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$

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Abstract

In this paper we present the algorithm of finding the solutions of the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$.

AMS Subject Classification: 17Sxx.
Key words: *p-adic numbers, solvability of equation, congruence.*

1 Introduction

In the present time description of different structures of mathematics are studying over field of p-adic numbers. In particular, now p-adic analysis is one of intensive developing directions of mathematics. Numerous applications of p-adic numbers found their own reflection in the theory of p-adic differential equations, p-adic theory of probabilities, p-adic mathematical physics and others.

In the field of complex numbers it is well known fundamental Abel's theorem about insolvability in radicals of general equation of n-th degree ($n > 5$). In this field square equation is solved by discriminant, for cubic equation there exist Cardano's formulas. In the field of p-adic numbers square equation no always has solution. It is known the criteria of solvability of the equation $x^2 = a$ [3] and in [1] we can find the solvability criteria for the equation $x^q = a$, where q is an arbitrary natural number.

The solvability criterion for the cubic equation $x^3 + ax = b$ in the field of $3-adic$ numbers is different from the case $p > 3$.

In [6] criteria of solvability for the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$ numbers with condition of $|a|_3 \neq \frac{1}{3}$ is studied.

Using the results of [6] in this paper we present the algorithm of finding the solutions of the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$ for any a.

2 Preliminaries

Let $\mathbb{Q}$ be the field of rational numbers. Every rational number $x \neq 0$ can be represented in the form $x = p^{\gamma(x)}\dfrac{n}{m}$, where $n, \gamma(x) \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$ and p is a fixed prime number. In the field $\mathbb{Q}$ we define a norm by

$$|x|_p = \begin{cases} p^{-\gamma(x)}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The norm $|x|_p$ is called a *p-adic norm* of x and it satisfies so called the strong triangle inequality. The completion of $\mathbb{Q}$ with respect to p-adic norm defines the

p-adic field which is denoted by $\mathbb{Q}_p$ ([3, 4]). It is well known that any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \ldots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p - 1$, $x_0 \neq 0$, $(j = 0, 1, \ldots)$. p-Adic numbers x, for which $|x|_p \leq 1$, are called *integer* p-adic numbers, and the set of these numbers is denoted by $\mathbb{Z}_p$. Integers $x \in \mathbb{Z}_p$, for which $|x|_p = 1$, are called *units* of $\mathbb{Z}_p$, and their set is denoted by $\mathbb{Z}_p^*$.

For any numbers a and m it is known the following

Theorem 1 [2]. *If* $(a, m) = 1$, *then a congruence* $ax \equiv b \,(mod\, m)$ *has one and only one solution.*

We also need the following

Lemma 1 [1]. *The following is true*:

$$\left(\sum_{i=0}^{\infty} x_i p^i\right)^q = x_0^q + \sum_{k=1}^{\infty} \left(q x_0^{q-1} x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right) p^k,$$

where $x_0 \neq 0$, $0 \leq x_j \leq p - 1$, $N_1 = 0$ and for $k \geq 2$

$$N_k = N_k(x_0, \ldots, x_{k-1}) = \sum_{\substack{m_0, m_1, \ldots, m_{k-1}: \\ \sum_{i=0}^{k-1} m_i = q, \quad \sum_{i=1}^{k-1} i m_i = k}} \frac{q!}{m_0! m_1! \ldots m_{k-1}!} x_0^{m_0} x_1^{m_1} \ldots x_{k-1}^{m_{k-1}}.$$

For $q = 3$ we have

$$\left(\sum_{i=0}^{\infty} x_i p^i\right)^3 = x_0^3 + \sum_{k=1}^{\infty} \left(3 x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right) p^k.$$

For $j \leq k$ we put

$$P_k^j = P_k^j(x_0, x_1, \ldots, x_{j-1}) = \sum_{\substack{m_0, m_1, \ldots, m_{j-1}: \\ \sum_{i=0}^{j-1} m_i = 3, \quad \sum_{i=1}^{j-1} i m_i = k}} \frac{6}{m_0! m_1! \ldots m_{j-1}!} x_0^{m_0} x_1^{m_1} \ldots x_{j-1}^{m_{j-1}}.$$

Also the following is true:

$$\left(\sum_{i=0}^{\infty} a_i p^i\right) \left(\sum_{j=0}^{\infty} x_j p^j\right) = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{k} x_s a_{k-s}\right) p^k. \tag{1}$$

3 The main result

Now let us study the main subject of our research – a cubic equation. Let we have the cubic equation

$$y^3 + r y^2 + s y + t = 0.$$

Note that, by replacing $y = x - \dfrac{r}{3}$, the cubic equation $y^3 + r y^2 + s y + t = 0$ is taken to the next equation

$$x^3 + a x = b. \tag{2}$$

So, we will study equation (2) in $\mathbb{Q}_p$, where $x = p^{\gamma(x)}(x_0 + x_1 p + \ldots)$, $a = p^{\gamma(a)}(a_0 + a_1 p + \ldots)$, $b = p^{\gamma(b)}(b_0 + b_1 p + \ldots)$, $x_j, a_j, b_j \in \{0, 1, \ldots, p-1\}$, $x_0, a_0, b_0 \neq 0$, $(j = 0, 1, \ldots)$.

In this paper we study the cubic equation over 3-adic numbers, i.e. $a, b \in \mathbb{Q}_3$ and $x \in \mathbb{Z}_3^*$.

Putting the canonical form of a, b and x in (2), we get

$$\left(\sum_{k=0}^{\infty} x_k 3^k\right)^3 + 3^{\gamma(a)} \left(\sum_{k=0}^{\infty} a_k 3^k\right)\left(\sum_{k=0}^{\infty} x_k 3^k\right) = 3^{\gamma(b)} \sum_{k=0}^{\infty} b_k 3^k.$$

By Lemma 1 and equality (1), the equation (2) becomes

$$x_0^3 + \sum_{k=1}^{\infty} \left(3x_0^2 x_k + N_k(x_0, x_1, \dots, x_{k-1})\right) 3^k +$$

$$+3^{\gamma(a)} \left(a_0 x_0 + \sum_{k=1}^{\infty} \left(\sum_{s=0}^{k} x_s a_{k-s}\right) 3^k\right) = 3^{\gamma(b)} \left(b_0 + \sum_{k=1}^{\infty} b_k 3^k\right). \qquad (3)$$

Proposition 1. *If one of the following conditions*:

$$1) \quad \gamma(a) = 0 \ \ and \ \ \gamma(b) < 0;$$

$$2) \quad \gamma(a) > 0 \ \ and \ \ \gamma(b) > 0;$$

$$3) \quad \gamma(a) > 0 \ \ and \ \ \gamma(b) < 0;$$

$$4) \quad \gamma(a) < 0 \ \ and \ \ \gamma(b) = 0;$$

$$5) \quad \gamma(a) < 0 \ \ and \ \ \gamma(b) > 0,$$

is fulfilled, then the equation (2) *has not a solution in* $\mathbb{Z}_3^*$.

Proof. 1) Let $\gamma(a) = 0$ and $\gamma(b) < 0$. Multiplying the equation (3) by $3^{-\gamma(b)}$, we get the following congruence

$$b_0 \equiv 0 \ \ (mod\, 3),$$

which is not correct. Consequently, the equation (2) has no solution in $\mathbb{Z}_3^*$.

2) Let $\gamma(a) > 0$ and $\gamma(b) > 0$. Then from (3) it follows a congruence

$$x_0^3 \equiv 0 \ \ (mod\, 3),$$

which has not a nonzero solution.Therefore, in $\mathbb{Z}_3^*$ the equation (2) has not a solution.

In other cases we analogously get the congruences

$$b_0 \equiv 0 \ \ (mod\, 3)$$

or

$$a_0 x_0 \equiv 0 \ \ (mod\, 3),$$

a contradiction. Therefore, in $\mathbb{Z}_3^*$ we have not a solution. $\square$

From the Proposition 1 we have that the cubic equation may have a solution if one of the following four cases

$$1) \ \gamma(a) = 0, \gamma(b) = 0, \quad 2) \ \gamma(a) = 0, \gamma(b) > 0,$$

$$3) \ \gamma(a) < 0, \gamma(b) < 0, \quad 4) \ \gamma(a) > 0, \gamma(b) = 0.$$

is hold.

The solvability criteria of the cubic equation for the cases 1), 2), 3) is given in [6]. The criteria for the case 4) $\gamma(a) > 0, \gamma(b) = 0$ is found only when $\gamma(a) > 1$, but

a problem of the finding the solvability criteria in the case of $\gamma(a) = 1, \gamma(b) = 0$ is open.

In this paper we present the algorithm of finding of the equation $x^3 + ax = b$ for all cases.

Theorem 2. *If $\gamma(a) = \gamma(b) = 0$ and $a_0 = 1$, then x to be a solution of the equation(2) in $\mathbb{Z}_3^*$ if and only if the next congruences*

$$x_0^3 + a_0x_0 \equiv b_0 \ (mod\, 3),$$

$$x_1a_0 + x_0a_1 + N_1(x_0) + M_1(x_0) \equiv b_1 \ (mod\, 3),$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1})+$$
$$+M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \ (mod\, 3), \quad k \geq 2$$

are fulfilled, where integers $M_k(x_0, \ldots, x_{k-1})$ are defined consequently from the following correlations

$$x_0^3 + a_0x_0 = b_0 + M_1(x_0) \cdot 3,$$
$$x_1a_0 + x_0a_1 + N_1(x_0) = b_1 - M_1(x_0) + M_2(x_0, x_1) \cdot 3,$$
$$x_{k-1}a_0 + x_{k-2}a_1 + \ldots + x_0a_{k-1} + x_0^2x_{k-2} + N_{k-1}(x_0, x_1, \ldots, x_{k-2}) =$$
$$= b_{k-1} - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + M_k(x_0, x_1, \ldots, x_{k-1}) \cdot 3, \quad k \geq 3.$$

Proof. Let the conditions of the theorem are given, then from [6] we have necessity and sufficiency of existence of a solution of the equation (2).

Let
$$x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + ..., \ \ 0 \leq x_j \leq 2, \ \ x_0 \neq 0, (j = 0, 1, ...)$$
– a solution of the equation (2), then equality (3) becomes

$$x_0^3 + \sum_{k=1}^{\infty} \left(3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right) 3^k +$$

$$+a_0x_0 + \sum_{k=1}^{\infty} \left(\sum_{s=0}^{k} x_sa_{k-s}\right) 3^k = b_0 + \sum_{k=1}^{\infty} b_k3^k.$$

So we have
$$x_0^3 + a_0x_0 + (x_1a_0 + x_0a_1 + N_1(x_0)) \cdot 3+$$
$$+\sum_{k=2}^{\infty} \left(x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1})\right) 3^k = b_0 + \sum_{k=1}^{\infty} b_k3^k,$$
from which it follows necessity of the fulfilling of the congruences of the theorem.

If the equation (2) has a solution $x \in \mathbb{Z}_3^*$, then from (3) it follows that

$$x_0^3 + a_0x_0 \equiv b_0 \ (mod\, 3).$$

Now let x is satisfied the congruences of the theorem. Since $(a_0, 3) = 1$, then by Theorem 1 there exist solutions x_k of the next congruences

$$x_0^3 + a_0x_0 \equiv b_0 \ (mod\, 3),$$

$$x_1a_0 + x_0a_1 + N_1(x_0) + M_1(x_0) \equiv b_1 \ (mod\, 3),$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1})+$$
$$+M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \ (mod\, 3), \quad k \geq 2,$$

where integers $M_k(x_0, \ldots, x_{k-1})$ are satisfied the conditions of the theorem.

Then

$$x_0^3 + \sum_{k=1}^{\infty} \left(3x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right) 3^k + a_0 x_0 + \sum_{k=1}^{\infty} \left(\sum_{s=0}^{k} x_s a_{k-s} \right) 3^k =$$

$$= x_0^3 + a_0 x_0 + (N_1 + x_0 a_1 + a_0 x_1)3+$$

$$+\sum_{k=2}^{\infty} \left(x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) + x_0 a_k + x_1 a_{k-1} + \ldots + x_{k-1} a_1 + x_k a_0\right) 3^k =$$

$$= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0, x_1) \cdot 3) \cdot 3+$$

$$+\sum_{k=2}^{\infty} (b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + M_{k+1}(x_0, x_1, \ldots, x_k) \cdot 3) \cdot 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k.$$

Therefore, we show that $x = \sum\limits_{k=0}^{\infty} x_k 3^k$ is a solution of the equation (2). $\square$

Let us examine a case $\gamma(a) = 0, \gamma(b) > 0$ and get necessary and sufficient conditions for a solution of the equation (2).

Theorem 3. *Let $\gamma(a) = 0$, $\gamma(b) = m > 0$ and $a_0 = 2$. Then x to be a solution of the equation(2) in $\mathbb{Z}_3^*$ if and only if the next congruences*

$$x_0^3 + a_0 x_0 \equiv 0 \,(mod\, 3),$$

$$x_1 a_0 + x_0 a_1 + N_1(x_0) + M_1(x_0) \equiv 0 \ (mod\, 3),$$

$$x_k a_0 + x_{k-1} a_1 + \ldots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1})+$$

$$+M_k(x_0, x_1, \ldots, x_{k-1}) \equiv 0 \ (mod\, 3), \ 2 \le k \le m-1,$$

$$x_k a_0 + x_{k-1} a_1 + \ldots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1})+$$

$$+M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_{k-m} \ (mod\, 3), \ k \ge m$$

are fulfilled, where integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined consequently from the following correlations

$$x_0^3 + 2x_0 = M_1(x_0) \cdot 3,$$

$$x_1 a_0 + x_0 a_1 + N_1(x_0) = -M_1(x_0) + 3M_2(x_0, x_1),$$

$$x_k a_0 + x_{k-1} a_1 + \ldots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) =$$

$$= -M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \ldots, x_k), \ 2 \le k \le m-1,$$

$$x_k a_0 + x_{k-1} a_1 + \ldots + x_0 a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) =$$

$$= b_{k-m} - M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \ldots, x_k), \ k \ge m.$$

Proof. From [6] we have necessity and sufficiency of existence of a solution of the equation (2).

Let

$$x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + ..., \ \ 0 \le x_j \le 2, \ \ x_0 \ne 0, (j = 0, 1, ...)$$

– a solution of the equation (2), then equality (3) becomes

$$x_0^3 + \sum_{k=1}^{\infty} \left(3x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right) 3^k+$$

$$+a_0x_0+\sum_{k=1}^{\infty}\left(\sum_{s=0}^{k}x_sa_{k-s}\right)3^k=3^m\left(b_0+\sum_{k=1}^{\infty}b_k3^k\right).$$

Therefore, we have

$$x_0^3+a_0x_0+(x_1a_0+x_0a_1+N_1(x_0))\cdot 3+$$

$$+\sum_{k=2}^{\infty}\left(x_ka_0+x_{k-1}a_1+\ldots+x_0a_k+x_0^2x_{k-1}+N_k(x_0,\ldots,x_{k-1})\right)3^k=$$

$$=3^m\left(b_0+\sum_{k=1}^{\infty}b_k3^k\right),$$

from which it follows necessity of the fulfilling of the congruences of the theorem.

Now let x is satisfied the congruences of the theorem. Since $(a_0,3)=1$, then by Theorem 1 there exist solutions x_k of the next congruences

$$x_0^3+a_0x_0\equiv 0\,(mod\,3),$$

$$x_1a_0+x_0a_1+N_1(x_0)+M_1(x_0)\equiv 0\ (mod\,3),$$

$$x_ka_0+x_{k-1}a_1+\ldots+x_0a_k+x_0^2x_{k-1}+N_k(x_0,x_1,\ldots,x_{k-1})+$$
$$+M_k(x_0,x_1,\ldots,x_{k-1})\equiv 0\ (mod\,3),\ 2\le k\le m-1,$$

$$x_ka_0+x_{k-1}a_1+\ldots+x_0a_k+x_0^2x_{k-1}+N_k(x_0,x_1,\ldots,x_{k-1})+$$
$$+M_k(x_0,x_1,\ldots,x_{k-1})\equiv b_{k-m}\ (mod\,3),\ k\ge m,$$

where integers $M_k(x_0,\ldots,x_{k-1})$ are satisfied the conditions of the theorem.

We have

$$x_0^3+\sum_{k=1}^{\infty}\left(3x_0^2x_k+N_k(x_0,x_1,\ldots,x_{k-1})\right)3^k+a_0x_0+\sum_{k=1}^{\infty}\left(\sum_{s=0}^{k}x_sa_{k-s}\right)3^k=$$

$$=x_0^3+a_0x_0+(N_1+x_0a_1+a_0x_1)3+$$

$$+\sum_{k=2}^{\infty}\left(x_0^2x_{k-1}+N_k(x_0,x_1,\ldots,x_{k-1})+x_0a_k+x_1a_{k-1}+\ldots+x_{k-1}a_1+x_ka_0\right)3^k=$$

$$=M_1(x_0)\cdot 3+(-M_1(x_0)+M_2(x_0,x_1)\cdot 3)\cdot 3+$$

$$+\sum_{k=2}^{m-1}(-M_k(x_0,x_1,\ldots,x_{k-1})+M_{k+1}(x_0,x_1,\ldots,x_k)\cdot 3)\cdot 3^k+$$

$$+\sum_{k=m}^{\infty}(b_{k-m}-M_k(x_0,x_1,\ldots,x_{k-1})+M_{k+1}(x_0,x_1,\ldots,x_k)\cdot 3)\cdot 3^k=3^m\left(b_0+\sum_{k=1}^{\infty}b_k3^k\right).$$

Therefore, we show that x is a solution of the equation (2). □

The following theorem gives necessary and sufficient conditions for a solution of the equation (2) for the case $\gamma(a)<0$ and $\gamma(b)<0$.

Theorem 4. *Let*

$$\gamma(a)=\gamma(b)=-m<0\ (m>0).$$

Then x to be a solution of the equation(2) in $\mathbb{Z}_3^*$ if and only if the next congruences

$$a_0x_0 \equiv b_0 \ (mod\, 3),$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \ (mod\, 3),\ 1 \leq k \leq m-1,$$

$$x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m + x_0^3 + M_m(x_0, x_1, \ldots, x_{m-1}) \equiv b_m \ (mod\, 3),$$

$$x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1} + M_{m+1}(x_0, x_1, \ldots, x_m) \equiv b_{m+1} \ (mod\, 3),$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) +$$
$$+ M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \ (mod\, 3),\ k \geq m+2$$

are fulfilled, where integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined consequently from the equalities

$$a_0x_0 = b_0 + M_1(x_0) \cdot 3,$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k = b_k - M_k(x_0, \ldots, x_{k-1}) + 3M_{k+1}(x_0, \ldots, x_k),\ 1 \leq k \leq m-1,$$

$$x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m + x_0^3 = b_m - M_m(x_0, x_1, \ldots, x_{m-1}) + 3M_{m+1}(x_0, x_1, \ldots, x_m),$$

$$x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1} = b_{m+1} - M_{m+1}(x_0, \ldots, x_m) + 3M_{m+2}(x_0, \ldots, x_{m+1}),$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) =$$
$$= b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \ldots, x_k),\ k \geq m+2.$$

Proof. Recall that the condition $\gamma(a) = \gamma(b) = -m < 0$ gives the solvability of the equation (2).

Multiplying (3) by 3^m, we get

$$3^m \left(x_0^3 + \sum_{k=1}^{\infty} \left(3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right) 3^k \right) + a_0x_0 + \sum_{k=1}^{\infty} \left(\sum_{s=0}^{k} x_s a_{k-s} \right) 3^k =$$

$$= a_0x_0 + \sum_{k=1}^{m-1} (x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k)\, 3^k +$$

$$+ (x_0^3 + x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m)3^m + (x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1})3^{m+1} +$$

$$+ \sum_{k=m+2}^{\infty} \left(x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) + x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k\right) 3^k =$$

$$= b_0 + \sum_{k=1}^{\infty} b_k 3^k,$$

which deduces necessity of the fulfilling of the congruences of the theorem.

Since $(a_0, 3) = 1$, then there exist unique solutions x_k of the next congruences

$$a_0x_0 \equiv b_0 \ (mod\, 3),$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \ (mod\, 3),\ 1 \leq k \leq m-1,$$

$$x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m + x_0^3 + M_m(x_0, x_1, \ldots, x_{m-1}) \equiv b_m \ (mod\, 3),$$

$$x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1} + M_{m+1}(x_0, x_1, \ldots, x_m) \equiv b_{m+1} \ (mod\, 3),$$

$$x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) +$$
$$+ M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \ (mod\, 3),\ k \geq m+2,$$

where integers $M_k(x_0, x_1, \ldots, x_{k-1})$, are defined as in the statement of the theorem.

Then

$$a_0x_0 + \sum_{k=1}^{m-1} (x_k a_0 + x_{k-1}a_1 + \ldots + x_0 a_k)\, 3^k +$$

$$+(x_0^3 + x_m a_0 + x_{m-1}a_1 + \ldots + x_0 a_m)3^m + (x_{m+1}a_0 + x_m a_1 + \ldots + x_0 a_{m+1})3^{m+1} +$$

$$+ \sum_{k=m+2}^{\infty} \left(x_0^2 x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) + x_k a_0 + x_{k-1}a_1 + \ldots + x_0 a_k\right) 3^k =$$

$$= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0, x_1) \cdot 3) \cdot 3 +$$

$$+ \sum_{k=2}^{\infty} (b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + M_{k+1}(x_0, x_1, \ldots, x_k) \cdot 3) \cdot 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k.$$

Therefore, we show that $x = \sum_{k=0}^{\infty} x_k 3^k$ is a solution of the equation (2). □

Examining various cases of $\gamma(a)$ and $\gamma(b)$ we need to study only the case $\gamma(a) > 0$ and $\gamma(b) = 0$. Because of appearance of uncertainty of a solution, we divide this case to $\gamma(a) > 1$ and $\gamma(a) = 1$.

Theorem 5. *Let $\gamma(a) = 2$, $\gamma(b) = 0$ and $(b_0, b_1) = (1, 0)$ or $(2, 2)$. Then x to be a solution of the equation (2) in $\mathbb{Z}_3^*$ if and only if he next congruences*

$$x_0^3 \equiv b_0 (mod\, 3),$$

$$x_0^3 \equiv b_0 + b_1 \cdot 3\, (mod\, 9),$$

$$x_0^2 x_1 + x_0 a_0 + M_1(x_0) \equiv b_2 (mod\ 3),$$

$$x_0^2 x_2 + P_3^2(x_0, x_1) + x_1 a_0 + x_0 a_1 + x_0 x_1^2 + M_2(x_0, x_1) \equiv b_3 (mod\ 3),$$

$$x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-2}a_0 + x_{k-3}a_1 + \ldots + x_0 a_{k-2} +$$
$$+ M_{k-1}(x_0, x_1, \ldots, x_{k-2}) \equiv b_k (mod\ 3), \quad k \geq 4$$

are fulfilled, where integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined from the equalities

$$x_0^3 = b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9,$$

$$x_0^2 x_1 + x_0 a_0 = b_2 - M_1(x_0) + 3M_2(x_0, x_1),$$

$$x_0^2 x_2 + P_3^2(x_0, x_1) + x_1 a_0 + x_0 a_1 + x_0 x_1^2 = b_3 - M_2(x_0, x_1) + 3M_3(x_0, x_1, x_2),$$

$$x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-2}a_0 + x_{k-3}a_1 + \ldots + x_0 a_{k-2} =$$
$$= b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1}), \quad k \geq 4.$$

Proof. Again from [6] we have necessity and sufficiency of existence of a solution of the equation (2).

If

$$x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + ..., \ \ 0 \leq x_j \leq 2, \ \ x_0 \neq 0, (j = 0, 1, ...)$$

– a solution of the equation (2), then equality (3) becomes

$$x_0^3 + \sum_{k=1}^{\infty} \left(3x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right) 3^k +$$

$$+3^m\left(a_0x_0+\sum_{k=1}^{\infty}\left(\sum_{s=0}^{k}x_sa_{k-s}\right)3^k\right)=b_0+b_1\cdot 3+\sum_{k=2}^{\infty}b_k3^k.$$

Since $N_k(x_0,x_1,\ldots,x_{k-1})$, $n\in\mathbb{N}$ depend only on $x_0,x_1,\ldots,x_{k-1}$, it is easy to check that $N_k(x_0,x_1,\ldots,x_{k-1})$ can be written

$$N_2(x_0,x_1)=3x_0x_1^2,$$

$$N_k(x_0,x_1,\ldots,x_{k-1})=P_k^{k-1}(x_0,x_1,\ldots,x_{k-2})+6x_0x_1x_{k-1},\quad k\geq 3,$$

and we have

$$x_0^3+x_0^2x_13^2+(x_0^2x_2+x_0x_1^2)3^3+\sum_{k=3}^{\infty}\left(3x_0^2x_k+P_k^{k-1}(x_0,x_1,\ldots,x_{k-2})+6x_0x_1x_{k-1}\right)3^k+$$

$$+3^m\left(a_0x_0+\sum_{k=1}^{\infty}\left(\sum_{s=0}^{k}x_sa_{k-s}\right)3^k\right)=b_0+b_1\cdot 3+\sum_{k=2}^{\infty}b_k3^k. \qquad (4)$$

Since $m=2$ the equality (4) becomes

$$x_0^3+(x_0^2x_1+x_0a_0)3^2+(x_0^2x_2+P_3^2(x_0,x_1)+x_1a_0+x_0a_1+x_0x_1^2)3^3+$$

$$+\sum_{k=4}^{\infty}\left(x_0^2x_{k-1}+P_k^{k-1}(x_0,\ldots,x_{k-2})+2x_0x_1x_{k-2}+x_{k-2}a_0+x_{k-3}a_1+\ldots+x_0a_{k-2}\right)3^k=$$

$$=b_0+b_1\cdot 3+\sum_{k=2}^{\infty}b_k3^k,$$

from which it follows necessity of the fulfilling of the congruences of the theorem.

Let x is satisfied the congruences of the theorem. Since $(a_0,3)=1$, then by Theorem 1 there exist solutions x_k of the next congruences

$$x_0^3\equiv b_0(mod\,3),$$

$$x_0^3\equiv b_0+b_1\cdot 3\,(mod\,9),$$

$$x_0^2x_1+x_0a_0+M_1(x_0)\equiv b_2(mod\ 3),$$

$$x_0^2x_2+P_3^2(x_0,x_1)+x_1a_0+x_0a_1+x_0x_1^2+M_2(x_0,x_1)\equiv b_3(mod\ 3),$$

$$x_0^2x_{k-1}+P_k^{k-1}(x_0,x_1,\ldots,x_{k-2})+2x_0x_1x_{k-2}+x_{k-2}a_0+x_{k-3}a_1+\ldots+x_0a_{k-2}+$$
$$+M_{k-1}(x_0,x_1,\ldots,x_{k-2})\equiv b_k(mod\ 3),\quad k\geq 4$$

where integers $M_k(x_0,x_1,\ldots,x_{k-1})$, are defined in the statement of the theorem.

So, we have

$$x_0^3+(x_0^2x_1+x_0a_0)3^2+(x_0^2x_2+P_3^2(x_0,x_1)+x_1a_0+x_0a_1+x_0x_1^2)3^3+$$

$$+\sum_{k=4}^{\infty}\left(x_0^2x_{k-1}+P_k^{k-1}(x_0,x_1,\ldots,x_{k-2})+2x_0x_1x_{k-2}+x_{k-2}a_0+x_{k-3}a_1+\ldots+x_0a_{k-2}\right)3^k=$$

$$=b_0+b_1\cdot 3+M_1(x_0)\cdot 9+(b_2-M_1(x_0)+3M_2(x_0,x_1))9+$$

$$+\sum_{k=3}^{\infty}(b_k-M_{k-1}(x_0,x_1,\ldots,x_{k-2})+3M_k(x_0,x_1,\ldots,x_{k-1}))\cdot 3^k$$

$$= b_0 + b_1 \cdot 3 + \sum_{k=2}^{\infty} b_k 3^k.$$

Therefore, we checked that x is a solution of the equation (2). $\square$

Theorem 6. *Let $\gamma(a) = 3$, $\gamma(b) = 0$ and $(b_0, b_1) = (1, 0)$ or $(2, 2)$. Then x to be a solution of the equation (2) in $\mathbb{Z}_3^*$ if and only if he next congruences*

$$x_0^3 \equiv b_0 (mod\, 3),$$

$$x_0^3 \equiv b_0 + b_1 \cdot 3\, (mod\, 9),$$

$$x_0^2 x_1 + M_1(x_0) \equiv b_2 (mod\ 3),$$

$$x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 a_0 + x_0 x_1^2 + M_2(x_0, x_1) \equiv b_3 (mod\ 3),$$

$$x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \ldots + x_0 a_{k-3} +$$
$$+ M_{k-1}(x_0, x_1, \ldots, x_{k-2}) \equiv b_k (mod\ 3), \quad k \geq 4$$

are fulfilled, where integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined from the equalities

$$x_0^3 = b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9,$$

$$x_0^2 x_1 = b_2 - M_1(x_0) + 3M_2(x_0, x_1),$$

$$x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 a_0 + x_0 x_1^2 = b_3 - M_2(x_0, x_1) + 3M_3(x_0, x_1, x_2),$$

$$x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \ldots + x_0 a_{k-3} =$$
$$= b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1}), \quad k \geq 4.$$

Proof. Analogously to the proof of the Theorem 5. $\square$

Similarly to the Theorem 5, it is proved the following

Theorem 7. *Let $\gamma(a) = m \geq 4$, $\gamma(b) = 0$ and $(b_0, b_1) = (1, 0)$ or $(2, 2)$. Then x to be a solution of the equation (2) in $\mathbb{Z}_3^*$ if and only if he next congruences*

$$x_0^3 \equiv b_0 (mod\, 3),$$

$$x_0^3 \equiv b_0 + b_1 \cdot 3\, (mod\, 9),$$

$$x_0^2 x_1 + M_1(x_0) \equiv b_2 (mod\ 3),$$

$$x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 x_1^2 + M_2(x_0, x_1) \equiv b_3 (mod\ 3),$$

$$x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} \equiv b_k (mod\ 3), \quad 4 \leq k \leq m-1,$$

$$x_0^2 x_{m-1} + P_m^{m-1}(x_0, x_1, \ldots, x_{m-2}) + 2x_0 x_1 x_{m-2} + x_0 a_0 \equiv b_m (mod\ 3),$$

$$x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-m} a_0 + \ldots + x_0 a_{k-m} \equiv b_k (mod\, 3), \quad k \geq m+1$$

are fulfilled, where integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined from the equalities

$$x_0^3 = b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9,$$

$$x_0^2 x_1 = b_2 - M_1(x_0) + 3M_2(x_0, x_1),$$

$$x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 x_1^2 = b_3 - M_2(x_0, x_1) + 3M_3(x_0, x_1, x_2),$$

$$x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} =$$
$$= b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1}), \quad 4 \leq k \leq m-1,$$

$$x_0^2x_{m-1}+P_m^{m-1}(x_0,\dots,x_{m-2})+2x_0x_1x_{m-2}+x_0a_0=$$
$$=b_m-M_{m-1}(x_0,\dots,x_{m-2})+3M_m(x_0,\dots,x_{m-1}),$$
$$x_0^2x_{k-1}+P_k^{k-1}(x_0,x_1,\dots,x_{k-2})+2x_0x_1x_{k-2}+x_{k-m}a_0+\dots+x_0a_{k-m}=$$
$$=b_k-M_{k-1}(x_0,x_1,\dots,x_{k-2})+3M_k(x_0,x_1,\dots,x_{k-1}),\quad k\geq m+1.$$

Now we consider the equality (3) with $\gamma(a)=1$, $\gamma(b)=0$. Then we get

$$x_0^3+\sum_{k=1}^{\infty}\left(3x_0^2x_k+N_k(x_0,x_1,\dots,x_{k-1})\right)3^k+3\left(a_0x_0+\sum_{k=1}^{\infty}\left(\sum_{s=0}^{k}x_sa_{k-s}\right)3^k\right)=$$
$$=x_0^3+a_0x_0\cdot 3+\sum_{k=2}^{\infty}\left(x_0^2x_{k-1}+N_k(x_0,x_1,\dots,x_{k-1})\right)3^k+$$
$$+\sum_{k=2}^{\infty}(x_{k-1}a_0+x_{k-2}a_1+\dots+x_0a_{k-1})\,3^k=b_0+\sum_{k=1}^{\infty}b_k3^k. \tag{5}$$

For $k\geq 1$, $s\geq 1$, $i\leq j-1$ we denote

$$A_0=x_0^2+a_0,$$
$$A_k=\frac{A_{k-1}}{3}+a_k+R_k,\ R_k=\sum_{j=0}^{k}x_jx_{k-j},\ k\geq 1,$$
$$N'_j=\begin{cases}\dfrac{N_{j-1}}{3},\ j=3s-1,\\ \dfrac{N_{j-1}}{3}+x_{\frac{j}{3}}^3,\ j=3s,\\ \dfrac{N_{j-1}-x_{\frac{j-1}{3}}^3}{3},\ j=3s+1,\end{cases}$$
$$S_j^i=\begin{cases}\dfrac{P_{j-1}^i}{3},\ j=3s-1,\\ \dfrac{P_{j-1}^i}{3}+x_{\frac{j}{3}}^3,\ j=3s,\\ \dfrac{P_{j-1}^i-x_{\frac{j-1}{3}}^3}{3},\ j=3s+1.\end{cases}$$

Theorem 8. *Let $\gamma(a)=1$, $\gamma(b)=0$ and $x\in\mathbb{Z}_3^*$ to be so that $A_0=x_0^2+a_0\not\equiv 0\,(mod\,3)$. Then x to be a solution of the equation (2) in $\mathbb{Z}_3^*$ if and only if the congruences*

$$x_0^3\equiv b_0\,(mod\,3),$$
$$x_0a_0+M_1(x_0)\equiv b_1\,(mod\,3),$$
$$(x_0^2+a_0)x_{k-1}+N'_k(x_0,x_1,\dots,x_{k-2})+x_{k-2}a_1+\dots+x_0a_{k-1}+$$
$$+M_k(x_0,\dots,x_{k-2})\equiv b_k\ (mod\ 3),\quad k\geq 2$$

are faithfully, where

$$M_1(x_0)=\frac{x_0^3-b_0}{3}$$

and integers $M_k(x_0,\dots,x_{k-2}),(k\geq 2)$ are defined from the equalities

$$x_0a_0+M_1(x_0)=b_1+M_2(x_0)\cdot 3,$$

$$(x_0^2+a_0)x_{k-1}+N'_k(x_0,x_1,\dots,x_{k-2})+x_{k-2}a_1+\dots+x_0a_{k-1}+$$
$$+M_k(x_0,\dots,x_{k-2})=b_k+M_{k+1}(x_0,\dots,x_{k-1})\cdot 3,\quad k\geq 2.$$

Proof. Let $x\in\mathbb{Z}_3^*$ be a solution of the equation (2), then we have

$$x_0^3+\sum_{k=1}^{\infty}\left(3x_0^2x_k+N_k(x_0,x_1,\dots,x_{k-1})\right)3^k+3\left(a_0x_0+\sum_{k=1}^{\infty}\left(\sum_{s=0}^{k}x_sa_{k-s}\right)3^k\right)=$$

$$=x_0^3+a_0x_0\cdot 3+\sum_{k=2}^{\infty}\left(x_0^2x_{k-1}+N_k(x_0,x_1,\dots,x_{k-1})\right)3^k+$$

$$+\sum_{k=2}^{\infty}(x_{k-1}a_0+x_{k-2}a_1+\dots+x_0a_{k-1})\,3^k=x_0^3+a_0x_0\cdot 3+$$

$$+\sum_{k=2}^{\infty}\left((x_0^2+a_0)x_{k-1}+N'_k(x_0,x_1,\dots,x_{k-2})+x_{k-2}a_1+\dots+x_0a_{k-1}\right)3^k=$$

$$=b_0+\sum_{k=1}^{\infty}b_k3^k.$$

Therefore, the congruences of the theorem are fulfilled.

Let a following system of the congruences

$$x_0^3\equiv b_0\,(mod\,3),$$

$$x_0a_0+M_1(x_0)\equiv b_1\,(mod\,3),$$

where $M_1(x_0)=\frac{x_0^3-b_0}{3}$, has a solution x_0. Then denote by $M_2(x_0)$ the number satisfying the equality $3M_2(x_0)=x_0a_0+M_1(x_0)-b_1$.

Using Theorem 1, we have existence of solutions x_k of the following congruences

$$(x_0^2+a_0)x_{k-1}+N'_k(x_0,x_1,\dots,x_{k-2})+x_{k-2}a_1+\dots+x_0a_{k-1}+$$
$$+M_k(x_0,\dots,x_{k-2})\equiv b_k\ (mod\ 3),\quad k\geq 2,$$

where integers $M_k(x_0,\dots,x_{k-2}),(k\geq 3)$ are defined from the equalities

$$(x_0^2+a_0)x_{k-1}+N'_k(x_0,x_1,\dots,x_{k-2})+x_{k-2}a_1+\dots+x_0a_{k-1}+$$
$$+M_k(x_0,\dots,x_{k-2})=b_k+M_{k+1}(x_0,\dots,x_{k-1})\cdot 3,\quad k\geq 2.$$

The next chain of equalities

$$x_0^3+a_0x_0\cdot 3+\sum_{k=2}^{\infty}\left((x_0^2+a_0)x_{k-1}+N'_k(x_0,x_1,\dots,x_{k-2})+x_{k-2}a_1+\dots+x_0a_{k-1}\right)3^k=$$

$$=b_0+M_1(x_0)\cdot 3+(b_1-M_1(x_0)+M_2(x_0)\cdot 3)\cdot 3+$$

$$+\sum_{k=2}^{\infty}(b_k-M_k(x_0,x_1,\dots,x_{k-2})+M_{k+1}(x_0,x_1,\dots,x_{k-1})\cdot 3)\cdot 3^k=b_0+\sum_{k=1}^{\infty}b_k3^k,$$

shows that x is a solution of the equation (2). $\square$

From the proof of Theorem 6 it is easy to see that if $A_0=x_0^2+a_0\equiv 0\,(mod\,3)$, then we have the following congruences and appropriate equalities

a) $x_0^3 \equiv b_0\,(mod\,3)$, i.e. $x_0^3 = b_0 + M_1(x_0)\cdot 3$;

b) $x_0 a_0 + M_1(x_0) \equiv b_1\,(mod\,3)$, then $x_0 a_0 + M_1(x_0) = b_1 + M_2(x_0)\cdot 3$;

c) $x_0 a_1 + M_2(x_0) \equiv b_2\,(mod\,3)$, then

$$x_0 a_1 + M_2(x_0) = b_2 + M_3(x_0)\cdot 3; \tag{6}$$

d) $\frac{A_0}{3}x_1 + x_1 a_1 + x_0 a_2 + x_0 x_1^2 + x_1^3 + M_3(x_0) \equiv b_3\,(mod\,3)$, it follows that

$$\frac{A_0}{3}x_1 + x_1 a_1 + x_0 a_2 + x_0 x_1^2 + x_1^3 + M_3(x_0) = b_3 + M_4(x_0, x_1)\cdot 3.$$

Since $A_1 = \frac{A_0}{3} + a_1 + 2x_0x_1$, then the congruence d) can be written in the form $(A_1 - x_0x_1)x_1 + x_0a_2 + x_1^3 + M_3(x_0) \equiv b_3\,(mod\,3)$, and so we have

$$(A_1 - x_0x_1)x_1 + x_0a_2 + x_1^3 + M_3(x_0) = b_3 + M_4(x_0, x_1)\cdot 3.$$

If for any natural number k we have $A_k \equiv 0\,(mod\,3)$, then we could establish the criteria of solvability for the equation (2). However, if there exists k, such that $A_k \not\equiv 0\,(mod\,3)$, then the criteria of solvability can be found, and therefore, we need the following

Lemma 2. *Let $\gamma(a) = 1$, $\gamma(b) = 0$ and $x \in \mathbb{Z}_3^*$ to be so that $A_{k-j} \equiv 0\,(mod\,3), 1 \le j \le k$, $A_k \not\equiv 0\,(mod\,3)$ for some fixed k. If x be a solution of the equation* (2), *then it is true the following system of the congruences*

$$x_0^3 \equiv b_0\,(mod\,3),$$

$$x_0a_0 + M_1(x_0) \equiv b_1\,(mod\,3),$$

$$x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^j + M_{2j}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j}(mod\,3), \tag{7}$$

$$(A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} + S_{2j+1}^j +$$
$$+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1}(mod\,3),$$

$$A_kx_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} + S_{2k+1+i}^{k+i} +$$
$$+ M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+1+i}(mod\,3),$$

where $1 \le j \le k$ and integers $M_k(x_0, \ldots, x_{k-2})$ are defined from the equalities

$$3\cdot M_1(x_0) = x_0^3 - b_0,$$

$$3\cdot M_2(x_0) = x_0a_0 + M_1(x_0) - b_1,$$

$$3\cdot M_{2j+1}(x_0, \ldots, x_{j-1}) = x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^j + M_{2j}(x_0, \ldots, x_{j-1}) - b_{2j},$$

$$3\cdot M_{2j+2}(x_0, x_1, \ldots, x_j) = (A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} +$$
$$+ S_{2j+1}^j + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) - b_{2j+1}, \tag{8}$$

$$3\cdot M_{2k+2+i}(x_0, x_1, \ldots, x_{k+i}) = A_kx_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} +$$
$$+ S_{2k+1+i}^{k+i} + M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) - b_{2k+1+i}.$$

Proof. We shall prove Theorem by induction. Let $k = 1$, i.e.

$$A_0 = x_0^2 + a_0 \equiv 0\,(mod\,3), \quad A_1 = \frac{A_0}{3} + a_1 + 2x_0x_1 \not\equiv 0\,(mod\,3),$$

then the system of the congruences (6) are true. Note that $S_2^1=0$, $S_3^1=x_1^3$.

From (5) it is easy to get

$$\frac{A_0}{3}x_{t-2}+x_{t-2}a_1+x_{t-3}a_2+\ldots+x_0a_{t-1}+S_t^{t-2}+2x_0x_1x_{t-2}+$$
$$+M_t(x_0,...,x_{t-3})\equiv b_t\,(mod\,p),\quad t\geq 4.$$

Therefore,

$$A_1x_{t-2}+x_{t-3}a_2+\ldots+x_0a_{t-1}+S_t^{t-2}+M_t(x_0,...,x_{t-3})\equiv b_t\,(mod\,3),\quad t\geq 4,\quad (9)$$

where

$$3\cdot M_{t+1}(x_0,...,x_{t-2})=A_1x_{t-2}+x_{t-3}a_2+\ldots+x_0a_{t-1}+S_t^{t-2}+M_t(x_0,...,x_{t-3})-b_t,\quad t\geq 4.$$

Obviously, the statement of Lemma is true for $k=1$, i.e. for $i=t-3$.

Let $k=2$, i.e. $A_0\equiv 0\,(mod\,3)$, $A_1\equiv 0\,(mod\,3)$ $A_2=\dfrac{A_1}{3}+a_2+x_1^2+2x_0x_2\not\equiv 0\,(mod\,3)$, then from the equalities (9) it follows that the following congruences are be added to the system (6):

e) $x_1a_2+x_0a_3+S_4^2+M_4(x_0,x_1)\equiv b_4\,(mod\,3)$, it follows

$$3\cdot M_5(x_0,x_1)=x_1a_2+x_0a_3+S_4^2+M_4(x_0,x_1)-b_4;$$

f) $\frac{A_1}{3}x_2+x_2a_2+x_1a_3+x_0a_4+S_5^3+M_5(x_0,x_1)\equiv b_5\,(mod\,3)$, it follows

$$3\cdot M_6(x_0,x_1,x_2)=\frac{A_1}{3}x_2+x_2a_2+x_1a_3+x_0a_4+S_5^3+M_5(x_0,x_1)-b_5;$$

$h)\dfrac{A_1}{3}x_{t-2}+x_{t-2}a_2+x_{t-3}a_3+\ldots+x_0a_t+S_{t+1}^{t-1}+M_{t+1}(x_0,x_1,\ldots,x_{t-3})\equiv b_{t+1}\,(mod\,3)$,

where $t\geq 5$ and $M_{t+2}(x_0,x_1,\ldots,x_{t-2})$ are defined by equalities

$$3\cdot M_{t+2}(x_0,x_1,\ldots,x_{t-2})=\frac{A_1}{3}x_{t-2}+x_{t-2}a_2+x_{t-3}a_3+\ldots+x_0a_t+$$
$$+S_{t+1}^{t-1}+M_{t+1}(x_0,x_1,\ldots,x_{t-3})-b_{t+1}.$$

Since $S_4^2=0$, $S_2^2=0$, $S_5^3=x_0x_2^2+x_1^2x_2$, $S_{t+1}^{t-1}=S_{t+1}^{t-2}+x_1^2x_{t-2}+2x_0x_2x_{t-2}$, we denote by $i=t-4$ and have

e) $x_1a_2+x_0a_3+M_4(x_0,x_1)\equiv b_4\,(mod\,3)$,

f) $(A_2-x_0x_2)x_2+x_1a_3+x_0a_4+M_5(x_0,x_1)\equiv b_5\,(mod\,3)$,

h) $A_2x_{i+2}+x_{i+1}a_3+x_ia_4+\ldots+x_0a_{i+4}+S_{i+5}^{i+2}+M_{i+5}(x_0,x_1,\ldots,x_{i+1})\equiv b_{i+5}\,(mod\,3)$, where

$$3\cdot M_5(x_0,x_1)=x_1a_2+x_0a_3+M_4(x_0,x_1)-b_4,$$
$$3\cdot M_6(x_0,x_1,x_2)=(A_2-x_0x_2)x_2+x_1a_3+x_0a_4+M_5(x_0,x_1)-b_5.$$
$$3\cdot M_{6+i}(x_0,x_1,\ldots,x_{i+2})=A_2x_{i+2}+x_{i+1}a_3+\ldots+x_0a_{i+4}+$$
$$+S_{i+5}^{i+2}+M_{i+5}(x_0,x_1,\ldots,x_{i+1})-b_{i+5}.$$

So we showed that the statement of Lemma is true for $k=2$.

Let the system of congruences (7)-(8) is true for k. By the induction hypothesis for $1 \leq j \leq k$ we have

$$x_0^3 \equiv b_0\,(mod\,3),$$

$$x_0 a_0 + M_1(x_0) \equiv b_1\,(mod\,3),$$

$$x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0 a_{2j-1} + S_{2j}^{j} + M_{2j}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j}(mod\,3),$$

$$(A_j - x_0 x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0 a_{2j} + S_{2j+1}^{j} +$$
$$+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1}(mod\,3).$$

Since $A_k \equiv 0(mod\,3)$, then from the congruences

$$A_k x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0 a_{2k+i} + S_{2k+1+i}^{k+i} +$$
$$+ M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+1+i}(mod\,3),\ i \geq 1$$

we derive

$$x_k a_{k+1} + x_{k-1}a_{k+2} + \ldots + x_0 a_{2k+1} + S_{2k+2}^{k+1} + M_{2k+2}(x_0, \ldots, x_k) \equiv b_{2k+2}(mod\,3),$$

$$\frac{A_k}{3}x_{k+1} + x_{k+1}a_{k+1} + x_k a_{k+2} + \ldots + x_0 a_{2k+2} + S_{2k+3}^{k+2} + M_{2k+3}(x_0, \ldots, x_k) \equiv b_{2k+3}(mod\,3),$$

$$\frac{A_k}{3}x_{k+1+i} + x_{k+1+i}a_{k+1} + x_{k+i}a_{k+2} + \ldots + x_0 a_{2k+i+2} + S_{2k+i+3}^{k+i+2} +$$
$$+ M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) \equiv b_{2k+i+3}(mod\,3), \quad i \geq 1.$$

It is easy to check that

$$S_{2k+3}^{k+2} = S_{2k+3}^{k+1} + R_{k+1}x_{k+1} - x_0 x_{k+1}^2,$$

$$S_{2k+i+3}^{k+i+2} = S_{2k+i+3}^{k+i+1} + R_{k+1}x_{k+1+i}, \quad i \geq 1.$$

By these correlations we deduce

$$\frac{A_k}{3}x_{k+1} + x_{k+1}a_{k+1} + x_k a_{k+2} + \ldots + x_0 a_{2k+2} + S_{2k+3}^{k+2} + M_{2k+3}(x_0, x_1, \ldots, x_k) =$$

$$= \frac{A_k}{3}x_{k+1} + x_{k+1}a_{k+1} + x_k a_{k+2} + \ldots + x_0 a_{2k+2} + S_{2k+3}^{k+1} + R_{k+1}x_{k+1} -$$

$$- x_0 x_{k+1}^2 + M_{2k+3}(x_0, x_1, \ldots, x_k) = (\frac{A_k}{3} + a_{k+1} + R_{k+1} - x_0 x_{k+1})x_{k+1} +$$

$$+ x_k a_{k+2} + \ldots + x_0 a_{2k+2} + S_{2k+3}^{k+1} + M_{2k+3}(x_0, x_1, \ldots, x_k) =$$

$$= (A_{k+1} - x_0 x_{k+1})x_{k+1} + x_k a_{k+2} + \ldots + x_0 a_{2k+2} + S_{2k+3}^{k+1} + M_{2k+3}(x_0, x_1, \ldots, x_k).$$

For $i \geq 1$ we get

$$\frac{A_k}{3}x_{k+1+i} + x_{k+1+i}a_{k+1} + \ldots + x_0 a_{2k+i+2} + S_{2k+i+3}^{k+i+2} + M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) =$$

$$= \frac{A_k}{3}x_{k+1+i} + x_{k+1+i}a_{k+1} + x_{k+i}a_{k+2} + \ldots + x_0 a_{2k+i+2} + S_{2k+i+3}^{k+i+1} + R_{k+1}x_{k+1} +$$

$$+ M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) = (\frac{A_k}{3} + a_{k+1} + R_{k+1})x_{k+1+i} +$$

$$+ x_{k+i}a_{k+2} + \ldots + x_0 a_{2k+i+2} + S_{2k+i+3}^{k+i+1} + M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) =$$

$$= A_{k+1}x_{k+1+i} + x_{k+i}a_{k+2} + \ldots + x_0a_{2k+i+2} + S^{k+i+1}_{2k+i+3} + M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}).$$

Consequently, we have

$$x_ka_{k+1} + x_{k-1}a_{k+2} + \ldots + x_0a_{2k+1} + S^{k+1}_{2(k+1)} + M_{2(k+1)}(x_0, \ldots, x_k) \equiv b_{2(k+1)}(mod\, 3),$$

$$(A_{k+1}-x_0x_{k+1})x_{k+1}+x_ka_{k+2}+\ldots+x_0a_{2k+2}+S^{k+1}_{2k+3}+M_{2k+3}(x_0, \ldots, x_k) \equiv b_{2k+3}(mod\, 3),$$

$$A_{k+1}x_{k+1+i} + x_{k+i}a_{k+2} + \ldots + x_0a_{2k+i+2} + S^{k+i+1}_{2k+i+3}+$$
$$+M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) \equiv b_{2k+i+3}(mod\, 3), \quad i \geq 1.$$

So we established that the system of congruences (7)-(8) is true for $k+1$. □

Theorem 9. *Let $\gamma(a) = 1$, $\gamma(b) = 0$ and $x \in \mathbb{Z}_3^*$ to be such that $A_{k-j} \equiv 0\,(mod\, 3), 1 \leq j \leq k$, $A_k \not\equiv 0\,(mod\, 3)$ for some fixed $k\,(k \geq 1)$. Then x to be a solution of the equation(2) in $\mathbb{Z}_3^*$ if and only if the system of the congruences*

$$x_0^3 \equiv b_0\,(mod\, 3),$$

$$x_0a_0 + M_1(x_0) \equiv b_1\,(mod\, 3),$$

$$x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S^j_{2j} + M_{2j}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j}(mod\, 3),$$

$$(A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} + S^j_{2j+1}+$$
$$+M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1}(mod\, 3),$$

has a solution, where $1 \leq j \leq k$ and integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined from the equalities

$$3 \cdot M_1(x_0) = x_0^3 - b_0,$$

$$3 \cdot M_2(x_0) = x_0a_0 + M_1(x_0) - b_1,$$

$$3{\cdot}M_{2j+1}(x_0, \ldots, x_{j-1}) = x_{j-1}a_j+x_{j-2}a_{j+1}+\ldots+x_0a_{2j-1}+S^j_{2j}+M_{2j}(x_0, \ldots, x_{j-1})-b_{2j},$$

$$3 \cdot M_{2j+2}(x_0, x_1, \ldots, x_j) = (A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j}+$$
$$+S^j_{2j+1} + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) - b_{2j+1}.$$

Proof. *Necessity.* If the equation (2) has a solution $x \in \mathbb{Z}_3^*$, i.e.

$$x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + ..., \quad 0 \leq x_j \leq 2,\ x_0 \neq 0,$$

then by Lemma 2 the system of the congruences (7) are true and $\{x_0, x_1, \ldots, x_k\}$ is a solution of this system.

Sufficiency. Let the system of the congruences (7) have a solution $x_0, x_1, \ldots, x_k$. Then by a condition $(A_k, 3) = 1$ and Theorem 1 we have an existence of solutions x_{k+i} of the following congruences

$$A_kx_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} + S^{k+i}_{2k+1+i}+$$
$$+M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+1+i}(mod\, 3),$$

where $i \geq 1$ and integers $M_{2k+i+2}(x_0, x_1, \ldots, x_{k+i})$ are defined recurrently from the following equalities

$$3 \cdot M_{2k+2+i}(x_0, x_1, \ldots, x_{k+i}) = A_kx_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i}+$$
$$+S^{k+i}_{2k+1+i} + M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) - b_{2k+1+i}.$$

Then we have

$$x_0^3 + a_0x_0 \cdot 3 + \sum_{j=1}^{k} \left(x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^j\right) 3^{2j} +$$

$$+ \sum_{j=1}^{k} \left((A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} + S_{2j+1}^j\right) 3^{2j+1} +$$

$$+ \sum_{i=1}^{\infty} \left(A_kx_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} + S_{2k+1+i}^{k+i}\right) \cdot 3^{2k+1+i} =$$

$$= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0) \cdot 3) \cdot 3 +$$

$$+ \sum_{j=1}^{k} (b_{2j} - M_{2j}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \cdot 3)\, 3^{2j} +$$

$$+ \sum_{j=1}^{k} (b_{2j+1} - M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+2}(x_0, x_1, \ldots, x_j) \cdot 3)\, 3^{2j+1} +$$

$$+ \sum_{i=1}^{\infty} (b_{2k+1+i} - M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) + M_{2k+2+i}(x_0, x_1, \ldots, x_{k+i}) \cdot 3)\, 3^{2k+1+i} =$$

$$= b_0 + \sum_{j=1}^{\infty} b_j 3^j.$$

So we checked that x, which is satisfied the conditions of the theorem, is a solution of the equation (2). $\square$

The next theorem complete the existence of a solution for the equation (2).

Theorem 10. *Let* $\gamma(a) = 1$, $\gamma(b) = 0$ *and* $x \in \mathbb{Z}_3^*$ *to be so that* $A_k \equiv 0\,(mod\,3)$ for all $k \in \mathbb{N}$. *Then* x *to be a solution of the equation*(2) *in* $\mathbb{Z}_3^*$ *if and only if the system of the congruences*

$$x_0^3 \equiv b_0\,(mod\,3),$$
$$x_0a_0 + M_1(x_0) \equiv b_1\,(mod\,3),$$
$$x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^j + M_{2j}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j}(mod\,3),$$
$$(A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} + S_{2j+1}^j +$$
$$+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1}(mod\,3),$$

has a solution, where $j \geq 1$ *and integers* $M_k(x_0, x_1, \ldots, x_{k-1})$ *are defined from the equalities*

$$3 \cdot M_1(x_0) = x_0^3 - b_0,$$
$$3 \cdot M_2(x_0) = x_0a_0 + M_1(x_0) - b_1,$$
$$3 \cdot M_{2j+1}(x_0, \ldots, x_{j-1}) = x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^j + M_{2j}(x_0, \ldots, x_{j-1}) - b_{2j},$$
$$3 \cdot M_{2j+2}(x_0, x_1, \ldots, x_j) = (A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} +$$
$$+ S_{2j+1}^j + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) - b_{2j+1}.$$

Proof. *Necessity.* If the equation (2) has a solution $x \in \mathbb{Z}_3^*$, i.e.

$$x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + ..., \quad 0 \leq x_j \leq 2,\ x_0 \neq 0,$$

then by Lemma 2 the system of the congruences (7) are true and $\{x_0, x_1, \ldots, x_k\}$ is a solution of this system.

Sufficiency. Let the system of the congruences (7) have a solution $x_0, x_1, \ldots, x_k$. Then by a condition $(A_k, 3) = 1$ and Theorem 1 we have an existence of solutions x_{k+i} of the following congruences

$$A_k x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0 a_{2k+i} + S^{k+i}_{2k+1+i} +$$
$$+M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+1+i}(mod\, 3),$$

where $i \geq 1$ and integers $M_{2k+i+2}(x_0, x_1, \ldots, x_{k+i})$ are defined recurrently from the following equalities

$$3 \cdot M_{2k+2+i}(x_0, x_1, \ldots, x_{k+i}) = A_k x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0 a_{2k+i} +$$
$$+S^{k+i}_{2k+1+i} + M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) - b_{2k+1+i}.$$

Then we have

$$x_0^3 + a_0 x_0 \cdot 3 + \sum_{j=1}^{\infty} \left(x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0 a_{2j-1} + S^j_{2j}\right) 3^{2j} +$$

$$+\sum_{j=1}^{\infty} \left((A_j - x_0 x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0 a_{2j} + S^j_{2j+1}\right) 3^{2j+1} =$$

$$= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0) \cdot 3) \cdot 3 +$$

$$+\sum_{j=1}^{\infty} (b_{2j} - M_{2j}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \cdot 3)\, 3^{2j} +$$

$$+\sum_{j=1}^{\infty} (b_{2j+1} - M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+2}(x_0, x_1, \ldots, x_j) \cdot 3)\, 3^{2j+1} =$$

$$= b_0 + \sum_{j=1}^{\infty} b_j 3^j,$$

consequently, x is a solution of the equation (2). □

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